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ASYMPTOTIC EXPANSIONS IN PROBABILITY THEORY

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1. Introduction

The natural solution to many problems in probability theory and mathematical statistics is provided by an examination of certain limit distributions. As is the case in the classical problem of the summation of a large number of random variables which are in some sense of equal weight, the study of the exact distribution functions of the sums not only leads, as a rule, to intractable formulas but in many important cases is impossible, since the exact distributions of the separate summands are often unknown. On the other hand, limit distributions are almost independent of the idiosyncrasies of the distributions of the summands and have a quite manageable form. The same phenomenon can be observed in mathematical statistics in the study of statistical criteria for a large number of observations. As a rule, the exact criteria are complicated but their limiting form is simple and convenient for application. An excellent example of this situation is the treatment of statistical mechanics by A. I. Khinchin, in which limit theorems play an outstanding role. Examples of this kind are literally innumerable.

However, as was pointed out a long time ago by P. L. Chebyshev [1], in order to be able to apply limit theorems in practice, it is necessary to have an estimate of the error involved. Obviously, if the remainder terms decrease slowly, then the limiting distributions must be used with corrections. Currently the most powerful and general method for finding corrections of this nature is the method of asymptotic expansions. These expansions were first examined, without an exact foundation, by Chebyshev [2] for the case of the classical limit theorem. Later, expansions of Chebyshev's type were studied by H. Bruns [3], C. Charlier [4], and F. Y. Edgeworth [5]. However, the fullest results in this direction were obtained much later by H. Cramér [6] and by C. G. Esseen [7]. We shall not concern ourselves here with the numerous further improvements in precision, since that is not the basic object of this paper. Nor shall we dwell on the series of interesting investigations of recent years, which studied asymptotic distributions for the sums of random variables and vectors forming a simple Markov chain. To a considerable extent, these works repeated methods which had been worked out for the sums of independent random variables and essentially consisted of the use of the properties of the Fourier transforms of functions of bounded variation.

The asymptotic analysis of the distributions of functionals of sums of random

variables began quite recently. The first result in this direction seems to be due to N. V. Smirnov [8]. This result is concerned with the speed of convergence of the distribution of the normed maximal difference between the true distribution function and the corresponding empirical distribution function, when the number of observations on which the latter is based is indefinitely increased. Let $F(x)$ be the distribution function of an observable random variable and assume that it is continuous. Let n be the number of independent observations and let $F_n(x)$ denote the corresponding empirical distribution function. Smirnov studied the random variable

$$(1) \qquad D_n^+ = \sqrt{n} \sup_{-\infty < x < \infty} [F_n(x) - F(x)]$$

and found the following facts [8].

For $0 < x < n^{1/2}$ we have

$$(2) \quad \Phi_n^+(x) = P\{D_n^+ < x\}$$

$$= 1 - \left(1 - \frac{x}{\sqrt{n}}\right)^n - x\sqrt{n} \sum_{k=1+[x\sqrt{n}]} \binom{n}{k} \frac{(k - x\sqrt{n})^k (n - k + x\sqrt{n})^{n-k-1}}{n^n}.$$

Also, we have $\Phi_n^+(x) = 0$ for $x < 0$ and $\Phi_n^+(x) = 1$ for $x > \sqrt{n}$. Another result of Smirnov is that, for all $x > 0$,

$$(3) \qquad \Phi^+(x) = \lim_{n \to \infty} \Phi_n^+(x) = 1 - e^{-2x^2}$$

and, for $0 < x < O(n^{1/6})$,

$$(4) \qquad \Phi_n^+(x) = 1 - e^{-2x^2}\left[1 + \frac{2x}{3\sqrt{n}} + O\left(\frac{1}{n}\right)\right].$$

Later, another term of the asymptotic expansion was found by N. I. Karplevskaia, a student at the University of Lvov. Her unpublished result (1949) is

$$(5) \qquad \Phi_n^+(x) = 1 - e^{-2x^2}\left[1 + \frac{2x}{3\sqrt{n}} + \frac{2x^2}{3n}\left(1 - \frac{2x^2}{3}\right) + O(n^{-3/2})\right].$$

Chan Li-Tsian added still another term [9] in this expansion. As a result, for $0 < x < O(n^{1/6})$, we have

$$(6) \qquad \Phi_n^+(x) = 1 - e^{-2x^2}$$

$$\left[1 + \frac{2x}{3\sqrt{n}} + \frac{2x^2}{3n}\left(1 - \frac{2x^2}{3}\right) + \frac{4x}{9n^{3/2}}\left(\frac{1}{5} - \frac{19x^2}{15} + \frac{2x^4}{3}\right) + O(n^{-2})\right].$$

A little later than Smirnov, and independently from him, Z. W. Birnbaum and F. H. Tingey [10] also found a result equivalent to (2), differing from it only in notation.

Using the exact distribution of the variable

$$(7) \qquad D_n = \sqrt{n} \sup_{-\infty < x < \infty} |F_n(x) - F(x)|$$

and performing elegant calculations, Chan Li-Tsian [11] succeeded in obtaining the expansion

$$(8) \qquad F_n(x) = P\{D_n < x\} = \sum_{r=0}^{3} n^{-r/2}K_r(x) + O(n^{-2}),$$

where the symbols K_r in formula (8) mean

$$K_0(x) = \sum_{k=-\infty} (-1)^k e^{-2k^2 x^2},$$

$$K_1(x) = -\frac{2x}{3} \sum_{k=-\infty} (-1)^k k^2 e^{-2k^2 x^2},$$

$$(9) \qquad K_2(x) = -\frac{1}{18} \sum_{k=-\infty} (-1)^k [f_1 - 4(f_1 + 3)k^2 x^2 + 8k^4 x^4] e^{-2k^2 x^2},$$

$$K_3(x) = \frac{x}{27} \sum_{k=-\infty}^{\infty} (-1)^k \left[\frac{f_2}{5} - \frac{4(f_2 + 45)k^2 x^2}{15} + 8k^4 x^4 \right] e^{-2k^2 x^2},$$

$$f_1 = k^2 \sin^2 \frac{k\pi}{2}, \qquad f_2 = 5k^2 + 22 - 15 \sin^2 \frac{k\pi}{2}.$$

The use of exact distributions in order to obtain asymptotic expansions is found in a note by B. V. Gnedenko [12], given to the study of the maximal deviations, both one-sided and two-sided, between two empirical distribution functions, each based on the same number of independent observations.

It is obvious that, given the exact distribution corresponding to a fixed value of the parameter n, it is always possible to obtain an asymptotic power-series expansion; success here depends only on the manipulative skill of the research worker. However, this method of obtaining asymptotic expansions can hardly be considered interesting mathematically. In the first place, each improvement in precision of the formula already obtained requires a repetition of the calculations performed earlier. In the second place, the method requires the knowledge of the exact distribution, and it is this last problem by itself that is likely to present considerable difficulties. Thus there is an acute need for a method that would permit us to find asymptotic expansions without first having to determine the exact distributions for all n.

In this particular respect the works of H. E. Daniels [13], I. I. Gikhman [14], [15], and V. S. Koroluk [16], [17], [18] are of undoubted interest, since not only did they solve separate particular problems but they also laid the foundation of a general method for obtaining asymptotic expansions. Daniels' article examined the asymptotic expansion of the probability density function of the arithmetic mean of identically distributed summands. Gikhman studied the asymptotic expansions of the expected value of sufficiently smooth functions of random variables forming a Markov chain. These expansions were obtained as a result of an analysis of the equation determining the required expectation. In particular, if the independent identically distributed random variables $\xi_1, \xi_2, \cdots, \xi_n$ have moments of the first three orders and the function $f(x)$ is uniformly continuous

and bounded, together with its derivatives up to the fifth order inclusive, then we have the equation

$$(10) \qquad E\left\{f\left(\frac{1}{n}\sum_{j=1}^{n}\xi_j\right)\right\} = f(a_1) + \frac{1}{2n}(a_2 - a_1^2)f''(a_1)$$

$$+ \frac{1}{n^2}\left\{\left[\frac{a_3 - a_1^3}{3!} - \frac{a_1(a_2 - a_1^2)}{2!}\right]f'''(a_1) + \frac{(a_2 - a_1^2)^2}{(2!)^3}f^{iv}(a_1)\right\} + o(n^{-2}),$$

where $a_k = E\{\xi_j^k\}$.

Koroluk studied the asymptotic expansions for the maximal deviations in a Bernoulli scheme using a recurrent system of differential equations. A mistake which slipped into papers [16] and [17] served to clarify the difficulties and pitfalls that threaten the research worker. It turned out that in his first works Koroluk did not take into account the effect of a jump through the limit, with the result that in the formulas he obtained several terms of the expansion were omitted. This was pointed out by A. N. Kolmogorov. His observations were used by A. A. Borovkov in [19] and [20] in the study of large deviations of the maximum of sums of independent, identically distributed, bounded lattice-like summands.

We limit ourselves to these few references to the existing literature, and make no claim to present an exhaustive study of the history of the question.

2. Random walks with assigned boundaries. Boundary layers

2.1. A number of fundamental problems of mathematical statistics and probability theory fit naturally into a scheme of random walk between some boundaries. In the case of random walk over a lattice, difference equations are obtained with some boundary conditions. The well-known book by W. Feller ([21], chapter 14) is an excellent introduction to this type of problem. In particular, it is known that the distribution of the above statistics D_n, introduced by Kolmogorov, is contained in this scheme. In addition to the determination of exact solutions for such problems, their asymptotic analysis is of considerable interest in the case when we assume that the variation of position of the particle in one step of the random walk will with overwhelming probability become infinitely small.

For the solution of problems of this type the following method can be suggested. The desired distribution is regarded as the solution of the equation

$$(11) \qquad P_\epsilon u_\epsilon = 0,$$

depending on the small parameter ϵ, which when $\epsilon \to 0$ degenerates into the differential equation

$$(12) \qquad L_0 u_0 = 0$$

of elliptic or parabolic type. In order to find the asymptotic expansion

$$(13) \qquad u_\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k u_k,$$

which is to be interpreted as meaning

$$(14) \qquad u_\epsilon - \sum_{k=0}^{N} \epsilon^k u_k = o(\epsilon^N),$$

methods can be used which were worked out since the times of Henri Poincaré in the theory of differential equations with a small parameter. This approach was employed in the previously mentioned papers by Gikhman and Koroluk. However, as we shall see presently, in the study of limit problems which arise from problems of random walks with absorbing or other types of barriers, the degeneration of equation (11) leads to the degeneration of the boundary conditions.

The usual process for the asymptotic solutions of equations with a small parameter consists of the expansion of the original operator P_ϵ into a series

$$(15) \qquad P_\epsilon = L_0 + \epsilon L_1 + \epsilon^2 L_2 + \cdots .$$

Then a formal solution of (11) is sought in the form of the series

$$(16) \qquad u_\epsilon = \sum_{k=0}^{\infty} \epsilon^k u_k.$$

By substituting this series into the original equation and comparing terms with the same power of ϵ, a recurrent system of equations is obtained that determines successively the terms of the asymptotic expansion

$$(17) \qquad \begin{aligned} L_0 u_0 &= 0, \\ L_0 u_m &= -\sum_{k=0}^{m-1} L_{m-k} u_k. \end{aligned}$$

Generally, the method just described is ineffective. The reason is that, while each successive step in the recurrence process reduces the inconsistencies involved in the earlier terms of the expansion, the inconsistencies in the fulfillment of boundary conditions remain untouched.

A similar situation was noted by M. I. Vishik and L. A. Liusternik in the asymptotic analysis of limit problems for differential equations in which higher order derivatives have "small" coefficients. Their results are summarized in the large paper [22]. The basic idea they used is as follows: insofar as the terms of a regular asymptotic expansion, defined by means of the system of equations (17), bring inconsistencies in the fulfillment of boundary conditions, it is essential to examine the complementary terms of the asymptotic expansion, which would act only near the boundary and would compensate for the inconsistencies in the fulfillment of boundary conditions. Liusternik and Vishik called these terms of the asymptotic expansion terms of the *boundary layer* type.

Recently the application of this idea to problems of probability theory has been the subject of studies of Koroluk [23], [24]. Naturally, the difference operators, the integral and the integro-differential operators P_ϵ become particularly important in this domain. The splitting of the original operator, which would warrant the second of the two processes mentioned, is more difficult here than for differential equations with small coefficients of higher derivatives. The general idea

just described will be illustrated with the example of a one-dimensional random walk on a lattice with step ϵ.

Let $u_\epsilon(x)$ denote the probability that the moving particle starting from $x > 0$ will reach the position $y \leq 0$ before visiting the region $y \geq 1$. Further, let p_k denote the probability of transition from x to $x + k\epsilon$ in a single step. For convenience in subsequent calculations, the probability that the moving particle will remain at x will be denoted by $1 + p_0$, with $p_0 < 0$ defined by $\sum_{-\infty}^{+\infty} p_k = 0$.

The required probability $u_k(x)$ satisfies the equation

$$(18) \qquad P_\epsilon u_\epsilon(x) = \sum_{-\infty}^{\infty} p_k u_\epsilon(x + k\epsilon) = 0,$$

subject to the boundary conditions

$$(19) \qquad u_\epsilon(-\epsilon r) = 1, \qquad u_\epsilon(1 + r\epsilon) = 0, \qquad\qquad r \geq 0.$$

We shall assume that the distribution p_k has $N + 3$ finite moments, for which we introduce the notation

$$(20) \qquad \epsilon \alpha_\epsilon = \sum k p_k, \qquad \beta_\epsilon = \frac{1}{2} \sum k^2 p_k, \qquad \gamma_{r\epsilon} = \frac{1}{r!} \sum k^r p_k,$$

for $3 \leq r \leq N + 3$. We shall assume further that the distribution p_k is $N + 3$ times differentiable with respect to ϵ, which yields the expansion

$$(21) \qquad p_k = p_{k0} + \sum_{r=1}^{N+2} \epsilon^r p_{kr} + \epsilon^{N+3} p_{k\epsilon}.$$

The above assumptions lead to the expansions

$$(22) \qquad \alpha_\epsilon = \alpha_0 + \sum_{s=1}^{N+2} \epsilon^s \alpha_s + \epsilon^{N+3} \alpha_{N+2,\epsilon},$$

$$(23) \qquad \beta_\epsilon = \beta_0 + \sum_{s=1}^{N+2} \beta^s \beta_s + \epsilon^{N+3} \beta_{N+3,\epsilon},$$

$$(24) \qquad \gamma_{r\epsilon} = \gamma_{r0} + \sum_{s=1}^{N+2} \epsilon^s \gamma_{rs} + \epsilon^{N+3} \gamma_{N+3,\epsilon}.$$

It is known (see [25], chapter 3) that, when $\epsilon \to 0$, the probability $u_\epsilon(x)$ tends to that solution $u_0(x)$ of the second order differential equation

$$(25) \qquad \beta_0 \frac{d^2 u}{dx^2} + \alpha_0 \frac{du}{dx} = 0,$$

which satisfies the boundary conditions

$$(26) \qquad u_0(0) = 1, \qquad u_0(1) = 0.$$

In this way, when $\epsilon \to 0$, the difference equation (18) degenerates into the differential equation (25) and, as can be seen easily by comparing (19) and (26), the boundary conditions are lost.

In accordance with the above general idea of the asymptotic method, the asymptotic series for the required probability $u_\epsilon(x)$ will contain *regular* asym-

totic terms (bounded and having bounded derivatives of all orders) and asymptotic terms of the *boundary layer* type, which decrease to zero at infinity (see [22]). Thus we shall seek an asymptotic expansion of the form

$$(27) \quad u_\epsilon(x) = \sum_{r=0}^{N} \epsilon^r u_r(x) + \epsilon \sum_{s=0}^{N+1} \epsilon^s \left[V_s^- \left(\frac{x}{\epsilon} \right) + V_s^+ \left(\frac{x-1}{\epsilon} \right) \right] + Z_\epsilon(x),$$

where $u_r(x)$ are the regular terms of the expansion, where V_s^- and V_s^+ are the boundary layers at the points $x = 0$ and $x = 1$, respectively, and Z_ϵ is the remainder term.

In order to determine the regular terms of the expansion, Taylor's formula and the expansions (22) to (24) are used to construct a recurrent system of differential equations. As a result, as indicated in (15), the operator P_ϵ on the class of infinitely differentiable functions is split into components of the type

$$(28) \quad \begin{aligned} L_0 &= \alpha_0 \frac{d}{dx} + \beta_0 \frac{d^2}{dx^2} \\ L_r &= \alpha_r \frac{d}{dx} + \beta_r \frac{d^2}{dx^2} + \sum_{s=1}^{r} \gamma_{s+2,r-s} \frac{d^{s+2}}{dx^{s+2}}, \qquad 1 \leqq r \leqq N. \end{aligned}$$

Thus, for the determination of the regular terms of the asymptotic expansion, we obtain a system of differential equations of the type (17), with the operator L_r defined by (28). The boundary conditions for the regular term u_r will be chosen so as to compensate for the inconsistencies of the boundary layers V_{r-1}^- and V_{r-1}^+ at the boundary points $x = 0$ and $x = 1$. Indeed we write

$$(29) \quad u_r(0) = -V_{r-1}^-(0), \qquad u_r(1) = -V_{r-1}^+(0).$$

The equations for the boundary layers are determined by a different process. After substituting (27) in (18) and carrying out the process for determining the regular terms of the expansion, we arrive at equation

$$(30) \quad \sum_{k=-\infty}^{\infty} p_k \left\{ \sum_{s=0}^{N+1} \epsilon^s \left[V_s^- \left(\frac{x}{\epsilon} + k \right) + V_s^+ \left(\frac{x-1}{\epsilon} + k \right) \right] + Z_\epsilon(x+k) \right\} = 0.$$

Using the asymptotic expansion for p_k we obtain from it equations determining V_r^- and V_r^+ by equating to zero the coefficients of the different powers of ϵ, separately for V_r^- and V_r^+. In the result V_r^- is determined as the solution of equations

$$(31) \quad P_0[V_0^-(y)] = \sum_{k=-\infty}^{\infty} p_{k0} V_0^-(y+k) = 0$$

and

$$(32) \quad P_0[V_r^-(y)] = -\sum_{l=1}^{r} \sum_{k=-\infty}^{\infty} p_{kl} V_{r-l}^-(y+k) = -\sum_{l=1}^{r} P_l[V_{r-l}^-(y)]$$

on the semiaxis $y \geqq 0$ that tends to zero at $+\infty$, and takes given values for $y < 0$. The meaning of the operators $P\hat{l}$ is clear from equations (31) and (32).

In exactly the same way V_r^+ is determined as the solution of equations

$$(33) \qquad P_0[V_0^+(y)] = \sum_{k=-\infty}^{\infty} p_{k0} V_0^+(y+k) = 0$$

and

$$(34) \qquad P_0[V_r^+(y)] = -\sum_{l=1}^{r} \sum_{k=-\infty}^{\infty} P_{kl} V_{r-l}^+(y+k) = -\sum_{l=1}^{r} P_l[V_{r-l}^+(y)]$$

on the semiaxis $y \leqq 0$ that takes given values when $y > 0$ and tends to zero when $y \to -\infty$.

We determine the boundary conditions for the boundary layers V_r^- in such a way that, in the fulfillment of conditions (19) when $y < 0$ by the function

$$(35) \qquad w_n^-(x) = \sum_{l=0}^{r} \epsilon^l u_l(x) + \sum_{l=0}^{r-1} \epsilon^{l+1} v_l^- \left(\frac{x}{\epsilon} \right)$$

the inconsistency should be of the order $O(\epsilon^{r+1})$. With this object in mind, the regular terms of the expansion are first extended in a continuous differentiable way beyond the limits of the interval $(0, 1)$. Let us consider for $t < 0$ the difference

$$(36)$$

$$w_N^-(\epsilon t) - 1 = u_0(\epsilon t) - 1 + \sum_{l=1}^{N} \epsilon^l [u_l(\epsilon t) - v_{l-1}^-(0)] + \sum_{l=0}^{N} \epsilon^{l+1} [v_l^-(t) - v_l^-(0)].$$

By applying Taylor's formula to the regular terms and equating the coefficients of equal powers of ϵ, we obtain

$$(37) \qquad V_r^-(t) = V_r^-(0) - \sum_{s=1}^{r+1} \frac{t^s}{s!} \frac{\partial^s u_{r+1-s}(0)}{\partial x^s}.$$

The values of the boundary layers $V_r^+(t)$ when $t > 1$ are determined in an analogous way,

$$(38) \qquad V_r^+(t) = V_r^+(1) - \sum_{s=1}^{r+1} \frac{t^s}{s!} \frac{\partial^s u_{r+1-s}(1)}{\partial x^s}.$$

By direct verification we can now ascertain that the algorithm we have constructed leads, for the remainder term, to equation

$$(39) \qquad P_\epsilon[Z_\epsilon(x)] = \epsilon^{N+3} g(x),$$

where $g(x)$ is a function bounded in $(0, 1)$ and the boundary conditions for $Z_\epsilon(x)$ are of the order $O(\epsilon^{N+1})$. From this we conclude that

$$(40) \qquad Z_\epsilon(x) = O(\epsilon^{N+1}).$$

The boundary layers of equations (31) to (35) can be found using the factorization method developed by M. G. Krein [26] for integral equations on the half line. We assume

$$(41) \qquad \Phi_0(\lambda) = \sum_{-\infty}^{\infty} p_{k0} \lambda^k, \qquad\qquad |\lambda| = 1.$$

For the function

$$(42) \qquad q(\lambda) = \frac{\Phi_0(\lambda)}{\lambda - 2 + \lambda^2} = \sum_{-\infty}^{\infty} q_k \lambda^k,$$

factorization is possible on the circle $|\lambda| = 1$. In other words, there exists the representation

$$(43) \qquad\qquad q(\lambda) = q_+(\lambda)q_-(\lambda),$$

in which the functions $q_+(\lambda)$ and $q_-(\lambda)$ are analytic and nonzero inside and outside the unit circle, respectively. Thus

$$(44) \qquad\qquad q_+(\lambda) = \sum_{k=0}^{\infty} q_k^+ \lambda^k, \qquad q_-(\lambda) = \sum_{k=0}^{\infty} q_k^- \lambda^{-k}.$$

The representation (43) enables us to transform the equations for $V_r^-(t)$ and $V_r^+(t)$ into the form

$$(45) \qquad\qquad \sum_{k=0}^{\infty} q_k^- V_0^-(t-k) = 0, \qquad \sum_{k=0}^{\infty} q_k^+ V_0^+(t+k) = 0.$$

The boundary layers $V_r^-(t)$ are determined in the form of a sum

$$(46) \qquad\qquad V_r^-(t) = V_{r0}^-(t) + \sum_{k=0}^{\infty} kw_r^-(t+k),$$

where $w_r^-(t)$ is the solution of the nonhomogeneous equation

$$(47) \qquad\qquad \sum_{k=-\infty}^{\infty} q_k w_r^-(t+k) = -\sum_{l=1}^{r} P_l[V_{r-l}^-(t)]$$

on the semiaxis $t \geqq 0$ that is equal to zero when $t < 0$, and $V_{r0}^-(t)$ is the solution of equation (44) that, when $t < 0$, takes the values

$$(48) \qquad\qquad V_{r0}^-(t) = V_r^-(t) - \sum_{k=0}^{\infty} (k-t)w_r^-(t).$$

The functions $V_r^+(t)$ are determined in a similar way.

We omit here the details of the proof of the proposed method, since this requires deep involvement in analytical detail.

2.3. To illustrate the application of the proposed algorithm let us examine the case when the initial distribution of the size of the jumps does not depend on ϵ, and the mathematical expectation of the size of the jump is equal to zero. In this case, as can be calculated easily, the regular terms of the asymptotic expansion have the form

$$(49) \qquad\qquad u_0(x) = 1 - x, \qquad u_r(x) = a_r + b_r x, \qquad r = 1, 2, \cdots.$$

In accordance with (29),

$$(50) \qquad\qquad a_r = -V_{r-1}^-(0), \qquad a_r + b_r = -V_{r-1}^+(0).$$

As indicated above, the boundary values for the boundary layers are determined by the equations

$$(51) \qquad\qquad V_r^-(t) = V_r^-(0) - b_r t, \qquad V_r^+(t) = V_r^+(0) - b_r t.$$

Equations (45) give

$$(52) \qquad\qquad V_r^-(0) = b_r q_-, \qquad V_r^+(0) = b_r q_+,$$

where

$$(53) \qquad q_- = \frac{q'_-(1)}{q_-(1)}, \qquad q_+ = \frac{q'_+(1)}{q_+(1)}.$$

Simple calculations lead to the following regular part of the asymptotic expansion

$$(54) \qquad u_\epsilon(x) = 1 + \frac{\epsilon q_-}{1 - \epsilon(q_- - q_+)} - \frac{x}{1 - \epsilon(q_- - q_+)}.$$

This function gives the solution of the asymptotic problem considered, with accuracy up to any power of ϵ, valid within the interval $(0, 1)$. Outside of fixed neighborhoods of the points $x = 0$ and $x = 1$, the boundary layers tend to zero faster than any power of ϵ and, therefore, do not influence the asymptotic expansion. It is interesting that in the present case it is possible to avoid the necessity of determining the boundary layers.

It is known the determination of the distributions of the statistics D_n and D_n^+, referred to at the beginning of this paper, can be reduced to the solution of certain difference equations (see, for example, [24] and [27]). The algorithm just described was used in [24] to obtain the asymptotic expansions of the distributions of these statistics. Naturally, the result obtained coincides with that of Chan Li-Tsian [11] which we indicated in the introduction. We cannot dwell here on a detailed examination of the series of interesting special problems in probability theory to which the algorithm we have constructed is applicable. We intend to do this in other publications.

3. Rate of convergence

3.1. The following method may be used to estimate the rate of convergence of the distribution functions $F_n(x)$ to a limiting distribution function $F(x)$ or to obtain the asymptotic expansions of $F_n(x)$. A random variable ξ_n is constructed, possessing the distribution $F_n(x)$ and depending upon n in the simplest possible manner. Let us assume that we have succeeded in representing ξ_n in the form

$$(55) \qquad \xi_n = \xi + \alpha_n \eta_n$$

where, as $n \to \infty$, the coefficient $\alpha_n \to 0$ and the random variables η_n tend in probability to a random variable η. If $E\eta_n^2 \leq C$ then, for every function g, bounded and having two bounded derivatives,

$$(56) \qquad Eg(\xi_n) = \int g(x)dF_n(x) = Eg(\xi) + \alpha_n Eg'(\xi)\eta + o(\alpha_n).$$

In this manner, if the joint distribution of ξ and η is known, it is possible to obtain an asymptotic representation of the expected value of any sufficiently smooth function.

In some cases the representation (55) makes it possible to estimate the speed of convergence of $F_n(x)$ to $F(x)$. For this purpose use can be made of the fact that for any $h > 0$ we have the inequalities

(57) $P\{\xi < x - h\} - P\{|\alpha_n \eta_n| \geqq h\} < P\{\xi_n < x\}$

$$< P\{\xi < x + h\} + P\{|\alpha_n \eta_n| \geqq h\}.$$

This implies that

(58) $|P\{\xi_n < x\} - P\{\xi < x\}|$

$$\leqq P\{\xi < x + h\} - P\{\xi < x - h\} + 2P\{|\alpha_n \eta_n| \geqq h\}.$$

If at the point x the function $F(x)$ satisfies a Lipschitz condition with constant K and $E\phi(\eta_n) \leqq C$, then

(59) $$|F_n(x) - F(x)| \leqq 2Kh + \frac{2C}{\phi(h/\alpha_n)}.$$

The following lemma may be helpful in many cases in the study of convergence of sums of independent random variables to a Brownian process. Here ξ_n stands for a functional on the increasing sums and one tries to obtain the representation (59).

LEMMA.   *Let $\Phi(x)$ be a distribution function such that*

(60) $$\int x\Phi(x) = 0,$$

and let $w(t)$ be a Brownian process. Then there exists a random variable τ such that $w(\tau)$ has the distribution function $\Phi(x)$, the event $\{\tau > s\}$ is independent of $w(t) - w(s)$ when $t > s$, and $w(\tau) - w(s)$ does not vanish when $s < \tau$. If $\int x^2 d\Phi(x) < \infty$, then

(61) $$E\tau = \int x^2 d\Phi(x).$$

If $\int x^{2n} d\Phi(x) < \infty$, then $E\{\tau^n\} < \infty$. Finally, if there exists a constant $C > 0$ such that $\Phi(C) - \Phi(-C) = 1$, then $|w(s)| \leqq C$ when $s < \tau$.

PROOF.   It is sufficient to give the argument only for the case when the distribution $\Phi(x)$ is discrete. First let $\Phi(x)$ have only the two points of growth, $-x_1$ and x_2 with jumps p_1 and p_2. By assumption $x_1 p_1 + x_2 p_2 = 0$. We shall denote by τ the moment at which the process $w(t)$, starting at the point 0 first reaches the boundary of the interval (x_1, x_2). Then $w(\tau)$ takes the values x_1 and x_2. Since $Ew(t) = 0$, the probabilities $\bar{p}_1$ and $\bar{p}_2$, with which $w(\tau)$ takes the corresponding values x_1 and x_2, satisfy the equations $\bar{p}_1 + \bar{p}_2 = 1$ and $x_1 \bar{p}_1 + x_2 \bar{p}_2 = 0$. This yields $\bar{p}_1 = p_1$ and $\bar{p}_2 = p_2$.

The fact that the event $\{\tau > s\}$ is independent of $w(t) - w(s)$ follows from the fact that the event $\{\tau > s\}$ is completely determined by the behavior of $w(t)$ for $t \leqq s$. It is easy to show that

(62) $$p_1 = \frac{x_2}{x_2 - x_1}, \qquad p_2 = -\frac{x_1}{x_2 - x_1},$$

and, consequently, that

(63) $$E\tau = -x_1 x_2 = x_1^2 p_1 + x_2^2 p_2.$$

Now let $\Phi(x)$ denote an arbitrary discrete distribution with $\int xd\Phi = 0$. This distribution $\Phi(x)$ can be represented as a mixture of discrete distributions $\Phi_k(x)$, each possessing only two points of growth, $x_1^{(k)}$ and $x_2^{(k)}$, for which $\int xd\Phi_k = 0$. In other words, we can write

$$(64) \qquad \Phi(x) = \sum_{k=1}^{\infty} \lambda_k \Phi_k(x),$$

where $\sum \lambda_k = 1$ and $\lambda_k \geqq 0$ for all k. In this case we construct the random variable τ by the following method. First, among the positive integers we choose at random one, the probability of choosing k being λ_k. If the integer selected is m, we choose τ to be equal to the value of t at which $w(t)$ would first reach the boundary of the interval $(x_1^{(m)}, x_2^{(m)})$. We omit the proof that in this case also the random variable τ satisfies all the conclusions of the lemma. In the general case, the proof of the lemma is achieved by approximating $\Phi(x)$ by a discrete distribution function.

In order to illustrate the applications of the lemma, we shall consider a sequence of independent random variables $\xi_1, \xi_2, \cdots, \xi_n$ and a sequence of corresponding distribution functions $\Phi_1(x), \Phi_2(x), \cdots, \Phi_n(x)$. We shall assume that for each $j = 1, 2, \cdots, n$ the expectation $E\xi_j = 0$. As before, let $w(t)$ stand for the Brownian motion process. Now, we use the lemma and denote by τ_1 the random variable satisfying its conditions and such that $w(\tau_1)$ has the distribution function $\Phi_1(x)$.

For each $s > 0$ the process $w(s + \tau_1) - w(\tau_1)$ also will be a Brownian process and moreover will be independent of $w(\tau_1)$. Let τ_2 be a random variable satisfying the requirements of the lemma for the process $w(s + \tau_1) - w(\tau_1)$ and having the distribution function $\Phi_2(x)$. Then the difference $w(\tau_2 + \tau_1) - w(\tau_1)$ has the distribution $\Phi_2(x)$ and is independent of $w(\tau_1)$. We can construct the quantities $\tau_3, \cdots, \tau_n$ in a similar way. Thus the differences $w(\sum_{i=1}^{k+1} \tau_i) - w(\sum_{i=1}^{k} \tau_i)$ have the distributions $\Phi_k(x)$ and for different k are independent. In this way, instead of the sequence of sums

$$(65) \qquad s_1 = \xi_1, s_2 = \xi_1 + \xi_2, \cdots, s_n = \xi_1 + \xi_2 + \cdots + \xi_n,$$

we can study the sequence of values of a Brownian process at the points $\eta_1 = \tau_1, \eta_2 = \tau_1 + \tau_2, \cdots, \eta_n = \tau_1 + \tau_2 + \cdots + \tau_n$. If

$$(66) \qquad \text{Var}(\xi_i) = \sigma_i^2, \qquad E\xi_i^4 < \infty, \qquad \text{Var}(\tau_i) = \alpha_i,$$

then

$$(67) \qquad \eta_k = \sum_{i=1}^{k} \sigma_i^2 + \sum_{i=1}^{k} (\tau_i - E\tau_i) = \sum_{i=1}^{k} \sigma_i^2 + (\sum_{i=1}^{n} \alpha_i)^{1/2} \frac{\sum_{i=1}^{k} (\tau_i - E\tau_i)}{(\sum_{i=1}^{n} \alpha_i)^{1/2}}.$$

We write

$$(68) \qquad t_k = \sum_{i=1}^{k} \sigma_i^2, \qquad \zeta(t_k) = \frac{\sum_{i=1}^{k} (\tau_i - E\tau_i)}{(\sum_{i=1}^{n} \alpha_i)^{1/2}}, \qquad \epsilon_n = (\sum_{i=1}^{n} \alpha_i)^{1/2}.$$

In this notation we have

$$(69) \qquad w(\eta_k) = w[t_k + \epsilon_n \zeta(t_k)].$$

In the case when the quantities ξ_i are identically distributed

$$\text{Var}(\xi_i) = \frac{1}{n}, \qquad E\xi_i^4 = O\left(\frac{1}{n^2}\right)$$

$$(70)$$

$$w(\eta_k) = w\left[\frac{k}{n} + \frac{c}{\sqrt{n}} \zeta\left(\frac{k}{n}\right)\right],$$

where

$$(71) \qquad \text{Var}(\tau_i) = \frac{c^2}{n^2}, \qquad \zeta\left(\frac{k}{n}\right) = \frac{\sqrt{n}}{c} \sum_{i=1}^{n} (\tau_i - E\tau_i).$$

Since $\xi(k/n)$ is finite, for sufficiently large n we have

$$(72) \qquad w(\eta_n) \approx w\left(\frac{k}{n}\right).$$

This circumstance can be used to obtain estimates of the rate of convergence and to find asymptotic expansions for the distribution of functionals defined on the successive sums of independent random variables.

3.2. Let us now turn to the consideration of two concrete problems.

3.2.1. THEOREM. *Let the random variables* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent and such that*

$$(73) \qquad E\xi_i = 0, \qquad \sum_{i=1}^{n} \text{Var}(\xi_i) = 1, \qquad P\{|\xi_i| > \delta_n\} = 0.$$

Further, let the functions $g_1(t)$ *and* $g_2(t)$ *be such that*

$$(74) \qquad g_1(t) < g_2(t), \qquad g_1(0) < 0 < g_2(0),$$

and such that for a certain k and for any t and h > 0 we have the inequalities

$$(75) \qquad |g_1(t+h) - g_1(t)| \leq kh, \qquad |g_2(t+h) - g_2(t)| \leq kh.$$

Let $S_k = \xi_1 + \xi_2 + \cdots + \xi_k$ *and*

$$(76) \qquad P_n = P\{g_1(t_i) < S_i < g_2(t_i); i = 1, 2, \cdots, n\}.$$

Under the conditions indicated there exists a number L, dependent only on k, $g_1(0)$, and $g_2(0)$, such that

$$(77) \qquad |P_n - P\{g_1(t) < w(t) < g_2(t); t \in [0, 1]\}|$$

$$\leq L\left[\delta_n + \max_i \text{Var}(\xi_i) + \gamma_n \log \frac{1}{\gamma_n}\right],$$

where $\gamma_n = \sum_{k=1}^{n} E\xi_k^4.$

PROOF.   In accordance with (69)

$$(78) \qquad P_n = P\{g_1(t_i) < w[t_i + \epsilon_n \xi(t_i)] < g_2(t_i); i = 1, 2, \cdots, n\}.$$

Since $|\xi_i| \leqq \delta_n$, when

$$(79) \qquad s \in [t_i + \epsilon_n \zeta(t_i), t_{i+1} + \epsilon_n \zeta(t_{i+1})]$$

by virtue of the lemma,

$$(80) \qquad |w(s) - w[t_i + \epsilon_n \zeta(t_i)]| < \delta_n.$$

Therefore

$$(81) \qquad P \begin{cases} g_1(t_i) + \delta_n + k(t_{i+1} - t_i) < w(s) < g_2(t_i) - \delta_n - k(t_{i+1} - t_i); \\ s \in [t_i + \epsilon_n \xi(t_i), t_{i+1} + \epsilon_n \xi(t_{i+1})], i = 1, 2, \cdots, n \end{cases}$$

$$\leqq P_n$$

$$\leqq P \begin{cases} g_1(t_i) - \delta_n - k(t_{i+1} - t_i) < w(s) < g_2(t_i) + \delta_n + k(t_{i+1} - t_i); \\ s \in [t_i + \epsilon_n \xi(t_i), t_{i+1} + \epsilon_n \xi(t_{i+1})], i = 1, 2, \cdots, n \end{cases}$$

Since

$$(82) \qquad |g_j(s) - g_j(t_i)| \leqq k|s - t_i| \leqq k(t_{i+1} - t_i) + \epsilon_n \max |\zeta(t_i)|,$$

therefore

$$(83) \qquad P \begin{cases} g_1(s) + \delta_n + 2k \max_i (t_{i+1} - t_i) + \epsilon_{.} \max_i |\zeta(t_i)| < w(s) \\ \qquad < g_2(s) - \delta_n - 2k \max_i (t_{i+1} - t_i) - \epsilon_n \max_i |\zeta(t_i)| \end{cases}$$

$$< P_n$$

$$< P \begin{cases} g_1(s) - \delta_n - 2k \max_i (t_{i+1} - t_i) - \epsilon_n \max_i |\zeta(t_i)| < w(s) \\ \qquad < g_2(s) + \delta_n + 2k \max_i (t_{i+1} - t_i) + \epsilon_n \max_i |\zeta(t_i)| \end{cases}$$

Let us now note that

$$(84) \qquad P\{\max_i |\zeta(t_i)| > log\ \epsilon_n^{-1}\} \leqq 4P\{|\zeta(t_n)| > log\ \epsilon_n^{-1}\}.$$

It can be shown that, under the assumptions adopted, this probability turns out to be a quantity of the order of $\delta_n^6 + \epsilon_n$. Therefore

$$(85) \qquad P\{g_1(s) + h < w(s) < g_2(s) - h; s \leqq 1 + \nu_n\} - O(\delta_n^6 + \epsilon_n)$$

$$< P_n < P\{g_1(s) - h < w(s) < g_2(s) + h; s \leqq 1 - \nu_n\} + O(\delta_n^6 + \epsilon_n),$$

where   .

$$(86) \qquad \begin{aligned} h &= \delta_n + 2k \max (t_{i+1} - t_i) + \nu_n, \\ \nu_n &= \epsilon_n\ log\ \epsilon_n^{-1}. \end{aligned}$$

Insofar as $(1/a)w(a^2 u)$ has the same distribution as $w(u)$, we find

$$(87) \quad P\left\{\frac{g_1[u(1+\nu_n)]+h}{(1+\nu_n)^{1/2}} < w(u) < \frac{g_2[u(1+\nu_n)]-h}{(1+\nu_n)^{1/2}}, 0 \leqq u \leqq 1\right\} - O(\delta_n^6 + \epsilon_n)$$

$$< P_n < P\left\{\frac{g_1[u(1-\nu_n)]-h}{(1-\nu_n)^{1/2}} < w(u) < \frac{g_2[u(1-\nu_n)]+h}{(1-\nu_n)^{1/2}}; 0 \leqq u \leqq 1\right\}$$

$$+ O(\delta_n^6 + \epsilon_n).$$

This gives us

$$(88) \quad |P_n - P\{g_1(s) < w(s) < g_2(s); s \in [0,1]\}|$$

$$< O(\delta_n^6 + \epsilon_n) + P\{g_1(s) - h_1 < w(s) < g_2(s) + h_1; s \in [0,1]\}$$

$$- P\{g_1(s) + h_1 < w(s) < g_2(s) - h_1; s \in [0,1]\}$$

$$\leqq P\{|\sup\,[w(s) - g_1(s)]| < h_1\} + P\{|\sup\,[w(s) - g_2(s)]| < h_1\} + O(\delta_n^6 + \epsilon_n),$$

where

$$(89) \quad h_1 = O[\max\,(t_{i+1} - t_i) + \delta_n + \nu_n].$$

It can be shown that the quantities $\sup_s\,[w(s) - g_1(s)]$ and $\sup_s\,[w(s) - g_2(s)]$ have finite density. Since $\epsilon_n = O(\gamma_n)$ this implies our assertion.

Particularizing the proposition just proved we obtain the following theorem.

THEOREM. *If the random variables ξ_i are identically distributed, bounded and if $E\xi_i = 0$, then*

$$(90) \quad |P\left\{g_1\left(\frac{k}{n}\right) < \frac{s_k}{\sqrt{n}} < g_2\left(\frac{k}{n}\right); k = 1, 2, \cdots, n\right\}$$

$$- P\{g_1(t) < w(t) < g_2(t); t \in [0,1]\}| \leqq Ln^{-1/2}\,log\,n.$$

3.2.2. THEOREM. *Let $\xi_1, \xi_2, \cdots, \xi_n$ be identically distributed and independent random variables, with $E\xi_i = 0$, $\mathrm{Var}(\xi_i) = 1/n$, and $E\xi_i^4 = O(n^{-2})$. Further, let $f(t, x)$ be a sufficiently smooth function of its arguments and*

$$(91) \quad \eta_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}, s_k\right).$$

Then

$$(92) \quad E\varphi(\eta_r) = E\varphi\left\{\int_0^1 f[s, w(s)]ds\right\} + \frac{c}{n}\varphi'\left\{\int_0^1 f[s, w(s)]ds\right\}$$

$$\left\{\int_0^1 f_x'[t, w(t)]w_1(t)dt - \int_0^1 f[t, w(t)]dw_1(t) + f|1, w(1)|w_1(1)\right\} + o(n^{-1/2}),$$

where $w(t)$ and $w_1(t)$ are two independent Brownian processes and $\varphi(x)$ a bounded differentiable function.

In order to shorten the formulas, it is convenient to write

$$(93) \quad \psi_k = \psi_k(n) = \frac{k}{n} + \frac{c}{\sqrt{n}}\zeta_n\left(\frac{k}{n}\right).$$

Then it follows that the quantity

$$(94) \qquad \bar{\eta}_n = \frac{1}{n} \sum_{k=1}^{n} f\left[\frac{k}{n}, (w\psi_k)\right]$$

is distributed in the same way as η_n. We write

$$(95) \qquad \bar{\eta}_n = \frac{1}{n} \sum_{k=1}^{n} f[\psi_k, w(\psi_k)] - \frac{1}{n} \sum_{k=1}^{n} \left\{ f[\psi_k, w(\psi_k)] - f\left[\frac{k}{n}, w(\psi_k)\right] \right\}.$$

It is obvious that

$$(96) \qquad \frac{1}{n} \sum_{k=1}^{n} f[\psi_k, w(\psi_k)] = \sum_{k=0}^{n-1} \int_{\psi_k}^{\psi_{k+1}} f[\psi_k, w(\psi_k)] ds + \frac{1}{n} f(0, 0)$$

$$- \frac{1}{n} f\left\{ 1, w\left[1 + \frac{c}{\sqrt{n}} \zeta_n(1) \right] \right\} - cn^{-1/2} \sum_{k=1}^{n-1} f[\psi_k, w(\psi_k)]\left[\zeta_n\left(\frac{k+1}{n}\right) - \zeta_n\left(\frac{k}{n}\right) \right].$$

It is easy to verify that

$$(97) \qquad \sum_{k=0}^{n-1} \int_{\psi_k}^{\psi_{k+1}} f[\psi_k, w(\psi_k)] ds = \int_{0}^{1+cn^{-1/2}\zeta_n(1)} f(s, w(s)) ds + \alpha_n$$

where α_r is such that $E\{(n^{-1/2}\alpha_n)^2\} \to 0$ when $n \to \infty$.

Further,

$$(98) \qquad \frac{1}{n} \sum_{k=1}^{n} \left\{ f[\psi_k, w(\psi_k)] - f\left[\frac{k}{n}, w(\psi_k)\right] \right\}$$

$$= cn^{-3/2} \sum_{k=1}^{n} f'_t\left[\frac{k}{n} + \theta_{nk} \frac{c}{\sqrt{n}} \zeta_n\left(\frac{k}{n}\right), w(\psi_k)\right] \zeta_n\left(\frac{k}{n}\right),$$

where $0 < \theta_{nk} < 1$.

Thus

$$(99) \qquad \bar{\eta}_n = \int_{0}^{1+cn^{-1/2}\zeta_n(1)} f[s, w(s)] ds$$

$$- cn^{-1/2} \sum_{k=1}^{n} f[\psi_k, w(\psi_k)]\left[\zeta_n\left(\frac{k+1}{n}\right) - \zeta_n\left(\frac{k}{n}\right) \right]$$

$$+ cn^{-3/2} \sum_{k=1}^{n} f'_t\left[\frac{k}{n} + \theta_{nk} \frac{c}{\sqrt{n}} \zeta_n\left(\frac{k}{n}\right), w(\psi_k)\right] \zeta_n\left(\frac{k}{n}\right)$$

$$+ \alpha_n.$$

The joint distribution of $w(s)$ and $\zeta_n(s)$ converges to the joint distribution of a two-dimensional homogeneous Gaussian process $[w(s), w_1(s)]$ with independent increments, for which

$$(100) \qquad\qquad Ew(s) = Ew_1(s) = 0,$$

$$Ew^2(s) = Ew_1^2(s) = s,$$

$$E\{w(s)w_1(s)\} = \frac{-\mu_3 s}{3c},$$

where μ_3 is the third moment of the quantity ξ_1.

Hence we find

$$(101) \qquad \bar{\eta}_r = \int_0^1 f[s, w(s)]ds + o\left(\frac{1}{\sqrt{n}}\right)$$

and so, if $\varphi(x)$ is a bounded differentiable function, then

$$(102) \qquad E\varphi(\eta_n) = E\varphi\{\int_0^1 f[s, w(s)]ds\}$$

$$+ cn^{-1/2}E\varphi'\{\int_0^1 f[s, w(s)]ds\}\{\int_0^1 f'[t, w(t)]w_1(t)dt$$

$$- \int_0^1 f[t, w(t)]dw_1(t) + f[1, w(1)]w_1(1)\}$$

$$+ o(n^{-1/2}).$$

This concludes our account of certain methods which can be used for obtaining both asymptotic expansions and estimates of the rate of convergence for functionals derived from distributions depending on a parameter.

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